MATH NOMAD · UNDERGRADUATE EXPOSITION

What Row Reduction Remembers

The geometry of intersecting lines, equation spaces, and the general-rank picture

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ROW REDUCTION · AFFINE GEOMETRY · PROJECTIVE GEOMETRY · GRASSMANNIANS

Abstract. Row reduction remembers the row space, and nothing else. This article develops that principle for a pair of line equations in the plane, where the row space becomes a plane of equations and RREF selects a coordinate-adapted basis. The rank-two orbit space is canonically the projective completion $\mathbb{RP}^2$ of the solution plane: consistent systems give finite solution points, while inconsistent parallel systems give ideal points. The final section extends the picture to arbitrary consistent and inconsistent linear systems.

Contents

Good mathematics books reward repeated reading. When we return to a familiar text with more mathematical maturity, ideas that once appeared merely computational may reveal a richer structure. My first linear algebra textbook was Hoffman and Kunze's *Linear Algebra* [1]. On rereading its first chapter many years later, I found myself viewing its discussion of systems of linear equations and row reduction from perspectives that were not available to me on my first reading.

This note records some of those observations. Its central aim is to examine reduced row echelon form (RREF) from several points of view—computational, geometric, linear-algebraic, and group-theoretic—and to understand what information row reduction preserves and what it forgets. For simplicity, much of the discussion concerns a system of two linear equations in two variables over $\mathbb{R}$:

$$\alpha_1 x + \beta_1 y = \gamma_1,$$
$$\alpha_2 x + \beta_2 y = \gamma_2.$$

Even this elementary system admits several useful representations. An equation

$$\alpha x + \beta y = \gamma$$

may be regarded as the affine-linear function

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad f(x, y) = \alpha x + \beta y - \gamma.$$

As an affine-linear function, its coefficients may be recorded as $(\alpha, \beta, -\gamma) \in \mathbb{R}^3$. Geometrically, the equation determines the zero set

$$L = Z(f) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\},$$

which is an affine line provided that $(\alpha, \beta) \neq (0, 0)$. The augmented-matrix convention keeps the constant on the right-hand side, so the same equation is represented there by the row

$$\begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}.$$

Thus the two equations above may be viewed simultaneously as two affine-linear functions f_1, f_2, two lines

$$L_1 = Z(f_1), \qquad L_2 = Z(f_2),$$

two vectors in a three-dimensional space of affine equations, or two rows of an augmented matrix. Solving the system means finding the common zero set

$$Z(f_1) \cap Z(f_2) = L_1 \cap L_2.$$

Equivalently, define

$$A = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad c = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}.$$

The system then becomes

$$AX = c,$$

with augmented matrix

$$[A \mid c] = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}.$$

Moving among these representations will allow us to reinterpret row operations and RREF, and ultimately to ask: what does row reduction remember about the original system?

> **Guiding theorem.** Two matrices of the same size are row-equivalent if and only if they have the same row space [1, Ch. 1]. Reduced row echelon form is the unique canonical representative of that row space [4]. Thus row reduction remembers the row space and forgets the ordered generating basis entirely. In the full-rank two-row case, the forgotten choices form a four-dimensional $\mathrm{GL}_2(\mathbb{R})$-worth of data; reordering and rescaling are only the most visible special cases.

Example 0.1. The systems

$$\begin{cases} 2x + y = 7, \\ \ \ x - y = 2 \end{cases} \qquad \text{and} \qquad \begin{cases} \ \ x + y = 4, \\ 3x - y = 8 \end{cases}$$

have no equation in common, yet both reduce to

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Their common row-space invariant records the same intersection point $(3, 1)$, while forgetting which two lines were originally chosen through that point.

The rest of the article unfolds the geometric meaning of this statement. Sections 1–4 move from computation to the geometry of transvections. Sections 5–7 identify the fixed equation space, its GL_2-frames, and the projective pencil. Section 8 classifies degeneracies, Section 9 identifies the rank-two orbit space with a projective plane, and Section 10 gives the arbitrary-size generalisations. A compact geometric dictionary concludes the article.

SECTION 1

What is RREF geometrically?

Geometrically, the solution of a system is especially easy to read when its equations have the form

$$x = a, \qquad y = b.$$

The first equation is a vertical line, the second is a horizontal line, and their intersection is visibly the point (a, b). By contrast, the intersection point is usually not apparent from two equations such as $\alpha_1 x + \beta_1 y = \gamma_1$ and $\alpha_2 x + \beta_2 y = \gamma_2$.

The purpose of row reduction is to replace a system by an equivalent one from which its solution can be read more easily. Reduced row echelon form is the canonical form that accomplishes this. Its formal definition applies to an arbitrary matrix, whether or not that matrix is being interpreted as an augmented matrix.

Definition 1.1. An $m \times n$ matrix R is said to be in **reduced row echelon form**, or **RREF**, if

1. the first non-zero entry in each non-zero row of R is 1;
2. each column containing the leading 1 of a row has all its other entries equal to 0;
3. every zero row occurs below every non-zero row;
4. if the non-zero rows are rows $1, \dots, r$, and the leading 1 of row i occurs in position k_i, then

$$k_1 < k_2 < \cdots < k_r.$$

The leading 1 in a non-zero row is called a *pivot*.

An augmented row (α, β, γ) will be called *genuine* when $(\alpha, \beta) \neq (0, 0)$. Only a genuine row represents an affine line; the remaining rows represent either the tautology $0 = 0$ or a contradiction $0 = \gamma \neq 0$.

Definition 1.2. If the rows of a matrix M are $r_1, \dots, r_m$, its **row space** is

$$\mathrm{Row}(M) = \mathrm{span}\{r_1, \dots, r_m\},$$

and its **rank** is

$$\mathrm{rank}(M) = \dim \mathrm{Row}(M).$$

For a matrix in RREF, the nonzero rows are linearly independent because their pivots occur in distinct columns. Its rank is therefore the number of nonzero rows, equivalently the number of pivots. Since elementary row operations replace the rows by another generating set for the same row space, they preserve rank.

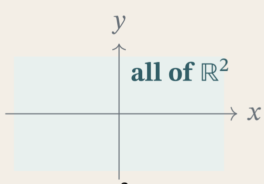
1.1 All 2×3 matrices in RREF

Let a 2×3 matrix be regarded as the augmented matrix of a system in the variables x and y, with the third entry of each row serving as the constant term.

Lemma 1.3. *Every 2×3 matrix in RREF has exactly one of the following seven forms. Here $a, b \in \mathbb{R}$.*

Rank 0

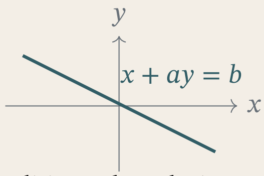
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Both equations are $0 = 0$, so they impose no restriction. Every point of $\mathbb{R}^2$ is a solution.

Rank 1

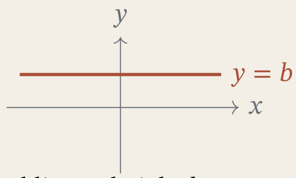
$$\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}$$



The nonzero row gives $x + ay = b$, and the zero row adds no condition. The solution set is one affine line. The sketch shows the generic case $a \neq 0$; when $a = 0$, the line is vertical.

Rank 1

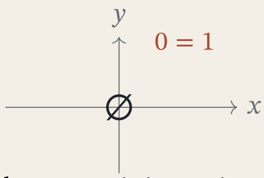
$$\begin{bmatrix} 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$



The nonzero row gives $y = b$. The solution set is the horizontal line at height b.

Rank 1

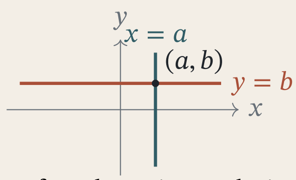
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



The nonzero row gives the impossible equation $0 = 1$. Hence the system is inconsistent and its solution set is empty.

Rank 2

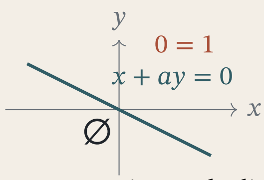
$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$



The equations are $x = a$ and $y = b$. Their intersection, and therefore the unique solution, is immediately visible as (a, b).

Rank 2

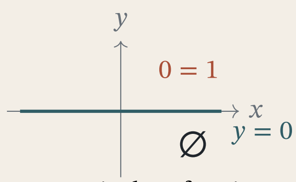
$$\begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



The first row defines the line $x + ay = 0$, but the second gives $0 = 1$. No point on the line satisfies the full system.

Rank 2

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



The first row defines $y = 0$, while the second gives $0 = 1$. The system is therefore inconsistent.

Proof of Lemma 1.3. The rank is 0, 1, or 2. There is only one matrix of rank zero, namely the zero matrix. A rank-one matrix has one pivot, which may occur in column 1, 2, or 3. Condition 3 forces the zero row to be the second row; the remaining RREF conditions determine the displayed form for each pivot position and leave only the entries to the right of the pivot free. A rank-two matrix has two pivots, whose positions must be one of

$$(1, 2), \qquad (1, 3), \qquad (2, 3).$$

Again the RREF conditions force the corresponding three displayed forms. These cases are mutually exclusive and exhaustive. □

Thus a consistent system of two equations in two variables has only three possible geometric solution sets in RREF: the whole plane, a line, or a single point. A pivot in the augmented position signals inconsistency.

1.2 How the RREF is obtained

The following finite procedure, usually called Gauss–Jordan elimination, transforms any matrix into RREF. The elimination tradition predates both names, and the Jordan associated with the reduced procedure is the German geodesist Wilhelm Jordan rather than Camille Jordan [2, 3].

1. Starting with the first row, find the leftmost column containing a nonzero entry in the rows not yet processed. If no such column exists, the procedure stops.
2. Interchange rows, if necessary, to move a nonzero entry in that column into the current row. This entry will be the next pivot.
3. Multiply the pivot row by the reciprocal of the pivot, making the pivot equal to 1.
4. Add suitable multiples of the pivot row to every other row, making every other entry in the pivot column equal to 0.
5. Move to the next row and repeat the procedure using only columns to the right of the previous pivot.

At each repetition the next pivot lies strictly below and strictly to the right of the preceding pivot. Since a matrix has only finitely many rows and columns, the procedure terminates after finitely many steps. The resulting matrix satisfies the four conditions in the definition of RREF.

Example 1.4. Consider the consistent system represented by

$$M = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 6 \end{bmatrix}.$$

We now follow the five steps of the Gauss–Jordan procedure described above.

Step 1: locate the leftmost available pivot column. The first column is the leftmost column containing a nonzero entry. Its nonzero entry is 2, which occurs in the second row.

Step 2: move a nonzero entry into the current pivot position. Interchange the two rows:

$$\begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}.$$

Step 3: normalise the pivot. The first pivot is 2. Dividing the first row by 2 makes it equal to 1:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_1 \mapsto \frac{1}{2} R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}.$$

Step 4: clear the rest of the pivot column. The other entry in the first pivot column is already 0. Thus this step produces no further change:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}.$$

Step 5: move to the next row and repeat. In the remaining submatrix, the next pivot is the entry 2 in the second column. Normalise it, and then clear the entry above it:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_2 \mapsto \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

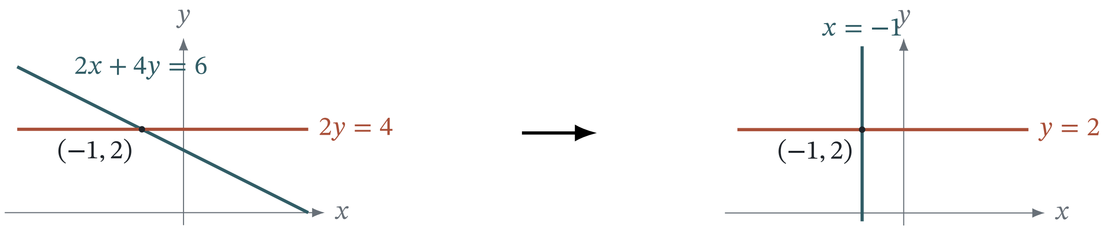
There are no further rows or columns to process and no zero rows to move. Therefore

$$\mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The RREF represents the equations

$$x = -1, \qquad y = 2.$$

Thus the original lines and the lines represented by the RREF have the same intersection point $(-1, 2)$.



1.3 How the row operations drive the algorithm

The three elementary row operations have distinct algorithmic roles. A row exchange moves an available nonzero entry into the desired pivot position. A nonzero row scaling normalises that pivot to 1. A transvection adds a multiple of the pivot row to another row and clears the remaining entries in the pivot column. Repeating these operations on the unprocessed rows produces the echelon staircase, with every new pivot strictly to the right of the preceding one.

Since there are only two rows and three columns, this process terminates after finitely many operations. Thus every 2×3 matrix can be transformed into its RREF by a finite sequence of row exchanges, row scalings, and transvections. Because every operation is invertible, all matrices appearing in the sequence are row-equivalent and represent systems with the same solution set.

Exchange and scaling are geometrically invisible in the affine plane; the transvection is where the visible geometry lives. Section 4 develops that motion, while Sections 3–5 identify the complete invariant preserved by all three operations.

SECTION 2

Cancellation selects the coordinate-parallel members

Return to the two line equations from the introduction and assume that

$$\Delta := \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0.$$

They meet at the unique point

$$P = A^{-1}c = (a, b).$$

For $t \in \mathbb{R}$, the transvection

$$f_2 \longmapsto f_2 + t f_1 \tag{1}$$

replaces the second equation by

$$(\alpha_2 + t\alpha_1)x + (\beta_2 + t\beta_1)y = \gamma_2 + t\gamma_1.$$

Every member of this family passes through P; together with the unchanged line $f_1 = 0$, these are the lines of the pencil through P. Among all lines through P, exactly one is horizontal and exactly one is vertical. Elimination finds them by cancelling coefficients.

To use L_1 to cancel the x-coefficient of L_2, one needs $\alpha_1 \neq 0$. If $\alpha_1 = 0$, then L_1 is itself horizontal—it is already the member $y = b$ of the pencil—and the two rows may be exchanged. Analogously, if $\beta_1 = 0$, then L_1 is already the vertical member $x = a$. A row interchange is therefore geometrically necessary rather than cosmetic: it selects a line capable of performing the required cancellation.

2.1 Clearing the x-coefficient

Assume $\alpha_1 \neq 0$. Choose $t = -\alpha_2/\alpha_1$ in Equation (1). Then the x-coefficient vanishes, while the remaining coefficient is

$$\beta_2 - \frac{\alpha_2}{\alpha_1}\beta_1 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1} = \frac{\Delta}{\alpha_1} \neq 0. \tag{2}$$

The right-hand side becomes

$$\gamma_2 - \frac{\alpha_2}{\alpha_1}\gamma_1 = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1}.$$

Thus the new equation is the genuine horizontal line

$$y = \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\Delta} = b. \tag{3}$$

2.2 Clearing the y-coefficient

If $\beta_1 \neq 0$, choose $t = -\beta_2/\beta_1$. Then the y-coefficient vanishes and

$$\alpha_2 - \frac{\beta_2}{\beta_1}\alpha_1 = -\frac{\Delta}{\beta_1} \neq 0. \tag{4}$$

The right-hand side becomes

$$\gamma_2 - \frac{\beta_2}{\beta_1}\gamma_1 = \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\beta_1},$$

so the resulting vertical line is

$$x = \frac{\beta_2\gamma_1 - \beta_1\gamma_2}{\Delta} = a. \tag{5}$$

Equations (3) and (5) are Cramer's rule, now read as the choice of the vertical and horizontal members of the pencil.

2.3 Gaussian elimination, with the example alongside

Suppose $\alpha_1 \neq 0$. Ordinary elimination performs

$$\begin{aligned}
R_2 &\longmapsto R_2 - \frac{\alpha_2}{\alpha_1}R_1 && \Rightarrow && \left[\,0 \quad \frac{\Delta}{\alpha_1} \,\middle|\, \frac{\alpha_1\gamma_2 - \alpha_2\gamma_1}{\alpha_1}\,\right],\\
R_2 &\longmapsto \frac{\alpha_1}{\Delta}R_2 && \Rightarrow && [\,0 \quad 1 \mid b\,],\\
R_1 &\longmapsto R_1 - \beta_1 R_2 && \Rightarrow && [\,\alpha_1 \quad 0 \mid \alpha_1 a\,],\\
R_1 &\longmapsto \frac{1}{\alpha_1}R_1 && \Rightarrow && [\,1 \quad 0 \mid a\,].
\end{aligned}$$

Geometrically: first replace one line by $y = b$, then use that horizontal line to replace the remaining line by $x = a$.

For

$$2x + y = 7, \qquad x - y = 2,$$

one may avoid fractions:

$$\begin{bmatrix} 2 & 1 & 7 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \mapsto 2R_2} \begin{bmatrix} 2 & 1 & 7 \\ 2 & -2 & 4 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - R_1} \begin{bmatrix} 2 & 1 & 7 \\ 0 & -3 & -3 \end{bmatrix}.$$

The second row is $y = 1$. Scaling it and eliminating the remaining y-coefficient gives

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - R_2} \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \mapsto \frac{1}{2} R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Hence $P = (3, 1)$. Notice that the combined-looking operation $R_2 \mapsto 2R_2 - R_1$ is not a fourth elementary operation: it is exactly the two legal operations displayed above. On the ordered pair of rows it is left multiplication by

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix},$$

whose determinant is 2, so the transformation is invertible. By contrast, replacing R_2 by $-R_1$ would correspond to the singular matrix

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

and would destroy information. Invertibility, rather than the superficial appearance of the formula, is what preserves the system.

For completeness,

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} \beta_2 & -\beta_1 \\ -\alpha_2 & \alpha_1 \end{bmatrix},$$

so left multiplication by A^{-1} produces $[I \mid (a, b)^{\mathrm{T}}]$ directly. This is the inverse-matrix form of the same geometric selection.

The calculation identifies what elimination selects, but not yet why such replacements preserve the original system. The next two sections supply that invariant explanation in the language of equation spaces and invertible row operations.

SECTION 3

Equations as vectors and the familiar row-space theorem

In the introduction, we associated several objects with the equation

$$\alpha x + \beta y = \gamma.$$

We now formalise those identifications. Let

$$\mathcal{A} = \{ f : \mathbb{R}^2 \to \mathbb{R} : f(x, y) = ux + vy + w \}$$

be the three-dimensional vector space of affine-linear functions. The augmented-matrix convention places the constant on the right-hand side, whereas the corresponding function is

$$f(x, y) = \alpha x + \beta y - \gamma.$$

The following linear isomorphism is the dictionary between those two conventions:

$$\Phi : \mathbb{R}^3 \longrightarrow \mathcal{A}, \qquad \Phi(\alpha, \beta, \gamma)(x, y) = \alpha x + \beta y - \gamma. \tag{6}$$

Thus the augmented row (α, β, γ) represents the affine-linear equation $\Phi(\alpha, \beta, \gamma) = 0$.

Two coordinate conventions.

Object	Coordinates	Vanishing at $P = (a, b)$
Augmented row	(α, β, γ)	$\alpha a + \beta b = \gamma$
Affine function $ux + vy + w$	(u, v, w)	$ua + vb + w = 0$
Dictionary Φ	$(\alpha, \beta, \gamma) \mapsto (\alpha, \beta, -\gamma)$	The two conditions agree.

The geometric representation is its zero set

$$Z(f) = \{Q \in \mathbb{R}^2 : f(Q) = 0\}.$$

When $(\alpha, \beta) \neq (0,0)$, this zero set is the affine line $\alpha x + \beta y = \gamma$. If $\alpha = \beta = 0$, the zero set is either all of $\mathbb{R}^2$, when $\gamma = 0$, or empty, when $\gamma \neq 0$.

For the system in the introduction, let $r_i = (\alpha_i, \beta_i, \gamma_i)$ be the rows of the augmented matrix M, and let

$$f_i = \Phi(r_i), \qquad L_i = Z(f_i), \qquad i = 1, 2.$$

The solutions of the system are precisely the points of $L_1 \cap L_2$.

If $E \in \mathrm{GL}_2(\mathbb{R})$, define $g_1, g_2 \in \mathcal{A}$ by

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Theorem 3.1. *The two pairs of functions span the same subspace of $\mathcal{A}$:*

$$\operatorname{span}\{f_1, f_2\} = \operatorname{span}\{g_1, g_2\}.$$

Moreover, for every point $Q \in \mathbb{R}^2$,

$$f_1(Q) = f_2(Q) = 0 \quad \Longleftrightarrow \quad g_1(Q) = g_2(Q) = 0.$$

Equivalently,

$$Z(f_1) \cap Z(f_2) = Z(g_1) \cap Z(g_2).$$

Proof. Each g_i is a linear combination of f_1, f_2, so

$$\operatorname{span}\{g_1, g_2\} \subseteq \operatorname{span}\{f_1, f_2\}.$$

Applying the same argument to

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = E^{-1} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

gives the reverse inclusion.

For $Q \in \mathbb{R}^2$, evaluation gives

$$\begin{bmatrix} g_1(Q) \\ g_2(Q) \end{bmatrix} = E \begin{bmatrix} f_1(Q) \\ f_2(Q) \end{bmatrix}.$$

Since E is invertible, the vector on the left is zero if and only if the vector on the right is zero. This is exactly the asserted pointwise equivalence, and hence the equality of the intersections of zero sets. $\square$

Elementary row operations are left multiplication by elementary invertible matrices. Therefore they preserve both the common solution set and the row space. Under Φ, the ordinary row space becomes

$$\Phi(\operatorname{Row} M) = \operatorname{span}\{f_1, f_2\} \subseteq \mathcal{A}. \tag{7}$$

Thus the usual statement that row operations preserve row space and the geometric statement that they preserve the simultaneous zero set are two consequences of the same invertible change of generators.

SECTION 4

The three row operations revisited geometrically

The motion created by a transvection looks rotation-like in the plane, but its true linearity lives in coefficient space. To reach that point, first observe that row exchange merely reverses the order in which the two equations are written, while row scaling replaces $f = 0$ by $\lambda f = 0$, where $\lambda \neq 0$. Neither operation changes a geometric line.

A transvection is different. It preserves the common solution set, but it generally replaces one of the two lines by a different line. This is the row operation with the richest geometric content, so it will be our focus.

Suppose

$$L_1 : \alpha_1 x + \beta_1 y = \gamma_1, \qquad L_2 : \alpha_2 x + \beta_2 y = \gamma_2$$

meet at the point $P = (a, b)$, and let

$$f_i(x, y) = \alpha_i x + \beta_i y - \gamma_i, \qquad i = 1, 2.$$

The transvection

$$R_2 \longmapsto R_2 + tR_1 \tag{8}$$

keeps f_1 fixed and replaces f_2 by

$$g_t = f_2 + t f_1. \tag{9}$$

Its zero set is

$$L_2(t) : \quad (\alpha_2 + t\alpha_1)x + (\beta_2 + t\beta_1)y = \gamma_2 + t\gamma_1. \tag{10}$$

Since $f_1(P) = f_2(P) = 0$, we have

$$g_t(P) = f_2(P) + t f_1(P) = 0.$$

Thus every line $L_2(t)$ passes through P. As t varies, the second line moves through the pencil of all lines centred at P, while L_1 remains fixed.

Assume $\Delta \neq 0$, so the two original lines meet in a single point. Then $L_2(t)$ is a genuine line for every t. Indeed, if both of its variable coefficients vanished, then

$$(\alpha_2, \beta_2) = -t(\alpha_1, \beta_1),$$

which would make the two original coefficient pairs proportional and force $\Delta = 0$.

When $\beta_2 + t\beta_1 \neq 0$, the slope of $L_2(t)$ is

$$m(t) = -\frac{\alpha_2 + t\alpha_1}{\beta_2 + t\beta_1}. \tag{11}$$

This is a fractional-linear function of t. Consequently, t is not the Euclidean angle through which the line has rotated. This is the sense in which the motion is linear only in the coefficients.

To make this precise, write the augmented rows as vectors

$$r_1 = (\alpha_1, \beta_1, \gamma_1), \qquad r_2 = (\alpha_2, \beta_2, \gamma_2)$$

in the augmented coefficient space $\mathbb{R}^3$. The transvection sends

$$r_2 \longmapsto r_2(t) = r_2 + t r_1.$$

Hence the moving row traces the affine line

$$\{r_2 + t r_1 : t \in \mathbb{R}\}$$

through r_2, with direction r_1, in $\mathbb{R}^3$. All three coefficients vary linearly:

$$\alpha_2(t) = \alpha_2 + t\alpha_1, \qquad \beta_2(t) = \beta_2 + t\beta_1, \qquad \gamma_2(t) = \gamma_2 + t\gamma_1.$$

The right-hand panel of Figure 1 displays only the first two coordinates of this motion. If

$$v_i = (\alpha_i, \beta_i) \in \mathbb{R}^2,$$

then the projected coefficient pair is

$$v_2(t) = v_2 + t v_1.$$

These points lie on a straight affine line through v_2, parallel to v_1. Passing from $v_2(t)$ to the slope of the corresponding geometric line requires taking the ratio in (11); this ratio is why linear motion in coefficient space appears nonlinear in the plane.

The plane picture makes transvection visible, while the coefficient-space picture makes it linear. The next section identifies the fixed vector space in which all these moving equations live.

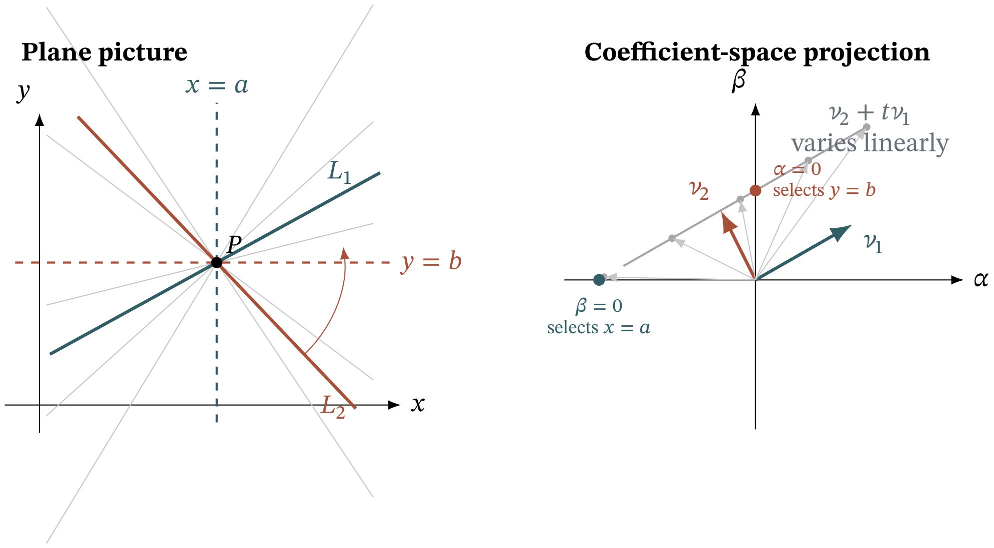


Figure 1: In the plane, transvection replaces L_2 by another line through P. In the (α, β)-plane, the moving coefficient pair follows the affine line $v_2 + tv_1$. Its crossings with $\alpha = 0$ and $\beta = 0$ select the horizontal and vertical members of the pencil. A usable pivot is precisely the transversality condition that makes the required crossing possible.

SECTION 5

The space V_P, the row space, and the complete invariant

Define

$$V_P = \{f \in \mathcal{A} : f(P) = 0\}. \tag{12}$$

This is a vector space because it is the kernel of the linear evaluation map

$$\mathrm{ev}_P : \mathcal{A} \to \mathbb{R}, \qquad \mathrm{ev}_P(f) = f(P).$$

The map is onto (the constant function 1 maps to 1), so rank–nullity gives $\dim V_P = 3 - 1 = 2$. More explicitly, if $P = (a, b)$, then

$$V_P = \{u(x-a) + v(y-b) : u, v \in \mathbb{R}\} = \mathrm{span}\{x-a, y-b\}. \tag{13}$$

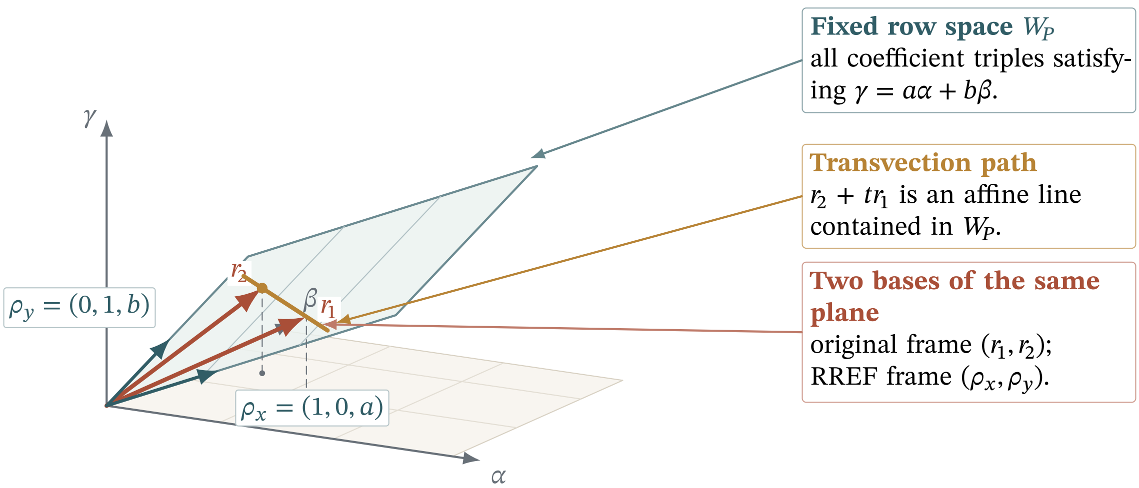


Figure 2: A three-dimensional view of augmented coefficient space. The rows may change, but their plane does not: the original frame (r_1, r_2), every point of the transvection path $r_2 + tr_1$, and the RREF frame $(\rho_x, \rho_y) = ((1, 0, a), (0, 1, b))$ all lie in the fixed plane $W_P \subset \mathbb{R}^3$. Dashed drop lines emphasise height in the γ-direction.

Proposition 5.1. *If L_1, L_2 meet uniquely at P, then*

$$\Phi(\operatorname{Row} M) = \operatorname{span}\{f_1, f_2\} = V_P.$$

Proof. Both f_1 and f_2 vanish at P, so their span lies in V_P. They are linearly independent: a relation $sf_1 + tf_2 = 0$ forces $s(\alpha_1, \beta_1) + t(\alpha_2, \beta_2) = (0, 0)$, and $\Delta \neq 0$ gives $s = t = 0$. Their span is therefore two-dimensional, as is V_P, so the inclusion is equality. □

Theorem 5.2. *Let $M = [A \mid c]$ be a 2×3 augmented matrix with $\det A \neq 0$, and let $P = (a, b)$ be its unique solution. Then*

$$\operatorname{rref}(M) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}.$$

Moreover, two such matrices are row-equivalent if and only if their systems have the same solution point.

Proof. The matrix M has rank 2, and row operations preserve both rank and consistency. Hence $\operatorname{rref}(M)$ has two nonzero rows and represents a consistent system. By Lemma 1.3, a rank-two 2×3 RREF has one of three forms:

$$\begin{bmatrix} 1 & 0 & d_1 \\ 0 & 1 & d_2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The last two contain the contradiction $0 = 1$, so consistency forces

$$\operatorname{rref}(M) = \begin{bmatrix} 1 & 0 & d_1 \\ 0 & 1 & d_2 \end{bmatrix}.$$

Its solution set is the singleton $\{(d_1, d_2)\}$. Row operations preserve the solution set, which was $\{P\}$; therefore $(d_1, d_2) = (a, b)$.

Now suppose M and N both have invertible coefficient blocks. If they have the same solution point P, then Proposition 5.1 gives

$$\Phi(\operatorname{Row} M) = V_P = \Phi(\operatorname{Row} N).$$

Since Φ is injective, $\operatorname{Row} M = \operatorname{Row} N$, and the two ordered row bases differ by an invertible change of basis; hence M and N are row-equivalent. Conversely, row-equivalent matrices have the same simultaneous zero set by Theorem 3.1, and therefore the same solution point. □

Figure 2 shows the same statement in augmented coefficient space: the rows move, but the plane does not.

SECTION 6

The GL_2-action on systems of line equations

The preceding sections identify the correct space on which the group acts. It is not the original xy-plane, but the space of ordered bases of the two-dimensional equation space associated with the intersection point. We continue to work over $\mathbb{R}$, although the arguments in this section hold over any field.

6.1 The coefficient plane and its ordered bases

Let $P = (a, b)$, and suppose $M = [A \mid c]$ has an invertible coefficient block and solution $P = A^{-1}c$. Define

$$W_P = \operatorname{Row}(M) \subseteq \mathbb{R}^3. \tag{14}$$

Every row combination represents an equation vanishing at P, so W_P is contained in

$$\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha a + \beta b = \gamma\} = \{(u, v, ua + vb) : u, v \in \mathbb{R}\}.$$

The displayed space is the kernel of the nonzero functional

$$(\alpha, \beta, \gamma) \longmapsto \alpha a + \beta b - \gamma,$$

and hence is two-dimensional. Since M has rank 2, so is $W_P = \operatorname{Row} M$; the inclusion is therefore equality. Under the isomorphism Φ from Equation (6),

$$\Phi(u, v, ua + vb) = u(x - a) + v(y - b),$$

so

$$\Phi(W_P) = V_P.$$

Thus W_P and V_P are two descriptions of the same two-dimensional vector space: one uses augmented coefficient rows, and the other uses affine-linear functions.

The rows of M form an ordered basis of W_P. Equivalently, (f_1, f_2) is an ordered basis of V_P. Write

$$\operatorname{Fr}(W_P) = \{(r_1, r_2) : r_1, r_2 \text{ form an ordered basis of } W_P\}$$

for the corresponding frame space.

6.2 The action on frames

For $E \in \mathrm{GL}_2(\mathbb{R})$, define

$$E \cdot M = EM.$$

If

$$E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix},$$

then the corresponding functions change according to

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

or

$$g_1 = e_{11} f_1 + e_{12} f_2, \qquad g_2 = e_{21} f_1 + e_{22} f_2.$$

By Theorem 3.1, the new pair has the same simultaneous zero set and spans the same equation space.

Proposition 6.1. *The action of* $\mathrm{GL}_2(\mathbb{R})$ *on* $\operatorname{Fr}(W_P)$ *is free and transitive. Equivalently, any two ordered bases of* W_P *are related by a unique element of* $\mathrm{GL}_2(\mathbb{R})$.

Proof. An ordered basis of W_P is equivalently a rank-two 2×3 matrix whose rows span W_P; under this identification the action on frames is exactly left multiplication on matrices.

Suppose $EM = M$. Since M has rank 2, it has a right inverse N, so $MN = I_2$. Therefore

$$EM = M \quad \Longrightarrow \quad EMN = MN \quad \Longrightarrow \quad E = I_2.$$

Thus the action is free.

Now let M and N have rows forming two ordered bases of the same plane W_P. Each row of N is a unique linear combination of the rows of M, so there is a unique 2×2 matrix E such that $N = EM$. Since the rows of N are independent, E is invertible. Hence the action is transitive. □

Once one reference basis is chosen, every other ordered basis is therefore represented by a unique element of $\mathrm{GL}_2(\mathbb{R})$. This is the group action directly used by row reduction.

6.3 The global orbit space and the RREF section

Consider all full-rank systems having a unique solution:

$$\mathcal{X} = \{[A \mid c] \in \mathrm{Mat}_{2\times 3}(\mathbb{R}) : A \in \mathrm{GL}_2(\mathbb{R})\}.$$

The group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathcal{X}$ by left multiplication. Define

$$q : \mathcal{X} \longrightarrow \mathbb{R}^2, \qquad q([A \mid c]) = A^{-1} c.$$

This map assigns to a system its intersection point.

Proposition 6.2. *Two matrices in $\mathcal{X}$ lie in the same $\mathrm{GL}_2(\mathbb{R})$-orbit if and only if they have the same intersection point. Consequently,*

$$\mathcal{X}/\,\mathrm{GL}_2(\mathbb{R}) \cong \mathbb{R}^2.$$

Proof. If $M = [A \mid c]$ and $E \in \mathrm{GL}_2(\mathbb{R})$, then

$$q(EM) = (EA)^{-1}Ec = A^{-1}c = q(M),$$

so q is constant on every orbit.

Conversely, suppose

$$M = [A \mid AP], \qquad N = [B \mid BP]$$

have the same solution point P. With $E = BA^{-1}$, we have $E \in \mathrm{GL}_2(\mathbb{R})$ and

$$EM = [B \mid BP] = N.$$

Thus the fibres of q are precisely the orbits. □

The row spaces of rank-two 2×3 matrices are points of the Grassmannian $\mathrm{Gr}(2,3)$. The systems in $\mathcal{X}$ correspond to the open chart consisting of planes that do not contain $(0,0,1)$. The following lemma identifies this chart precisely.

Lemma 6.3 (The unique-solution chart). *Let $M = [A \mid c]$ have augmented rank 2, let $W = \operatorname{Row} M$, and let $e_3 = (0,0,1)$. Then*

$$A \in \mathrm{GL}_2(\mathbb{R}) \iff e_3 \notin W.$$

In this case projection onto the first two coordinates restricts to an isomorphism $W \to \mathbb{R}^2$, and W is the graph of a unique linear functional $(u,v) \mapsto ua + vb$, namely $W = W_P$ for a unique $P = (a,b)$.

Proof. Projecting W onto the first two coordinates has image $\operatorname{Row} A$. Its kernel is $W \cap \operatorname{span}(e_3)$, so rank–nullity gives

$$\dim\bigl(W \cap \operatorname{span}(e_3)\bigr) = 2 - \operatorname{rank} A.$$

Thus the kernel is zero exactly when $\operatorname{rank} A = 2$. In that case the projection is an isomorphism, so W is the graph of a unique linear map $\mathbb{R}^2 \to \mathbb{R}$, which has the form $(u,v) \mapsto ua + vb$. If A is singular, the kernel is one-dimensional and therefore contains e_3. This proves both directions and the graph description. □

RREF chooses a distinguished representative in every orbit. Indeed,

$$A^{-1}[A \mid c] = [I_2 \mid A^{-1}c] = [I_2 \mid P].$$

Thus the map

$$s : \mathbb{R}^2 \longrightarrow \mathcal{X}, \qquad s(a,b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix},$$

is a section of the quotient map q. In this precise sense, RREF selects one canonical representative of each GL_2-orbit.

The dimensions make the loss of information tangible. The set $\mathcal{X}$ is open in $\mathbb{R}^6$, while $\dim \mathrm{GL}_2(\mathbb{R}) = 4$; because the action is free, the quotient has dimension $6 - 4 = 2$, exactly the dimension of the remembered solution plane.

6.4 Elementary matrices generate GL_2

The elementary row operations are represented by the matrices

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D_1(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}, \qquad D_2(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix},$$

and

$$U(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \qquad L(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},$$

where $\lambda \neq 0$. They represent row exchange, row scaling, and the two possible transvections. Their inverses are again elementary:

$$S^{-1} = S, \qquad D_i(\lambda)^{-1} = D_i(\lambda^{-1}), \qquad U(t)^{-1} = U(-t), \qquad L(t)^{-1} = L(-t).$$

Theorem 6.4. *Every matrix in* $\mathrm{GL}_2(\mathbb{R})$ *is a product of elementary row matrices.*

Proof. Let

$$G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \delta = \det G = ad - bc \neq 0.$$

If $a \neq 0$, direct multiplication gives the factorisation

$$G = L(c/a)\, D_1(a)\, D_2(\delta/a)\, U(b/a). \tag{15}$$

Every factor on the right is elementary.

If $a = 0$, then $\delta = -bc \neq 0$, so $c \neq 0$. The matrix SG has nonzero upper-left entry and therefore admits a factorisation of the displayed form. Since $S^{-1} = S$, the matrix $G = S(SG)$ is also a product of elementary matrices. □

The factorisation in (15) is the 2×2 *LDU* decomposition: the two middle scaling factors form the diagonal matrix D. When $a = 0$, the preliminary row exchange is the permutation in a *PLU* factorisation. Thus the same case split that generates GL_2 also explains why an elimination algorithm may need to exchange its rows before continuing.

Remark 6.5 (Row exchange is redundant as a generator). In fact, $\mathrm{GL}_2(\mathbb{R})$ is already generated by the diagonal scaling matrices and the transvections. The row-exchange matrix can be recovered from them through the factorisation

$$S = D_2(-1)\, U(1)\, L(-1)\, U(1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Reading the factors from right to left, a row interchange may therefore be performed using only transvections and a scaling:

$$R_1 \mapsto R_1 + R_2, \qquad R_2 \mapsto R_2 - R_1, \qquad R_1 \mapsto R_1 + R_2, \qquad R_2 \mapsto -R_2.$$

Here each operation uses the rows produced by the preceding operation. Thus the three familiar kinds of elementary row operation are convenient and geometrically distinct, but row exchange is not needed as an independent generator of $\mathrm{GL}_2(\mathbb{R})$.

The transvections $U(t)$ and $L(t)$ generate $\mathrm{SL}_2(\mathbb{R})$. Since every transvection has determinant 1, however, no product of transvections alone can equal S, whose determinant is -1; a scaling of determinant -1, such as $D_2(-1)$, is genuinely necessary. Over fields the same elementary-generation argument works verbatim, but over general commutative rings it can fail [7].

This theorem closes the gap between an invertible change of basis and an actual sequence of row operations. If two ordered pairs of equations through P are related by

$$N = EM, \qquad E \in \mathrm{GL}_2(\mathbb{R}),$$

then $E = E_k \cdots E_1$ for elementary matrices E_i, and hence

$$M \longmapsto E_1 M \longmapsto E_2 E_1 M \longmapsto \cdots \longmapsto E_k \cdots E_1 M = N$$

is a finite sequence of elementary row operations.

SECTION 7

The pencil, projective geometry, and coordinates

Classically, a pencil of concurrent lines is a projective line [5, 6]. A nonzero function $f \in V_P$ defines a line $Z(f)$ through P, but f and λf define the same line for every $\lambda \neq 0$. Consequently, the geometric pencil is not the vector space V_P itself but its projectivisation

$$\mathbb{P}(V_P) = (V_P \setminus \{0\})/\mathbb{R}^\times \cong \mathbb{RP}^1. \tag{16}$$

The family $[f_2 + tf_1]$, $t \in \mathbb{R}$, is one affine chart on this projective line; the missing member is $[f_1]$, naturally labelled $t = \infty$. Equation (11) is therefore a Möbius transformation between two projective coordinates: the row parameter t and the slope parameter.

Via $\Phi : W_P \to V_P$, the same pencil may be written as $\mathbb{P}(W_P)$. After choosing a basis, the linear group $\mathrm{GL}(W_P) \cong \mathrm{GL}_2(\mathbb{R})$ acts on nonzero equation vectors and hence on their projective classes:

$$T \cdot [r] = [T(r)].$$

A scalar matrix acts trivially on every projective class, so the effective group acting on the pencil is

$$\mathrm{PGL}(W_P) = \mathrm{GL}(W_P)/(\mathbb{R}^\times I) \cong \mathrm{PGL}_2(\mathbb{R}).$$

The three related actions should therefore be distinguished:

$$\begin{array}{ccl} \mathrm{GL}_2(\mathbb{R}) & \text{acts freely and transitively on} & \text{ordered bases of } W_P, \\ \mathrm{GL}(W_P) & \text{acts linearly on} & W_P, \\ \mathrm{PGL}(W_P) & \text{acts projectively on} & \mathbb{P}(W_P). \end{array}$$

The first action governs row reduction; the last governs the geometric pencil of individual lines through P.

There is a coordinate-free formulation. Let an affine plane be modelled on a two-dimensional direction space E. For fixed P, every covector $\lambda \in E^*$ defines

$$f_\lambda(Q) = \lambda(Q - P).$$

The map $E^* \to V_P$, $\lambda \mapsto f_\lambda$, is a linear isomorphism. No inner product is needed. If a Euclidean inner product is chosen, a covector may be represented by its normal vector, which is the optional picture used in Figure 1.

Now choose an ordered basis (e_1, e_2) of E, with dual basis (e^1, e^2). Then

$$Z(f_{e^1}) = P + \mathrm{span}(e_2), \qquad Z(f_{e^2}) = P + \mathrm{span}(e_1).$$

Thus the pencil contains exactly one member parallel to each basis direction, and RREF selects those two members. In ordinary coordinates they are $x = a$ and $y = b$.

This also explains the asymmetry between rows and columns. Row operations change the basis of the equation space V_P and preserve the solution set. Columns encode the chosen coordinate functions and their order. Swapping columns changes or relabels the coordinate frame; it is not a row equivalence unless the corresponding change of variables is explicitly tracked.

If arbitrary operations are allowed on both sides,

$$M \longmapsto EMF, \qquad E \in \mathrm{GL}_m(\mathbb{R}), \quad F \in \mathrm{GL}_n(\mathbb{R}),$$

then the complete invariant collapses to rank alone: every rank-r matrix is equivalent to a block matrix with I_r and zeros. Row equivalence therefore remembers a point of a Grassmannian, whereas full two-sided equivalence remembers only which Grassmannian the point lies in. This contrast is one reason the row-space viewpoint is richer than rank alone.

SECTION 8

Degeneracies and the rank criterion

8.1 Degenerate equations: tautology and contradiction

The expression "an equation with coefficients $0, 0$" means an equation of the form

$$0x + 0y = \gamma.$$

Such an equation is not an affine line. Under the correspondence

$$0x + 0y = \gamma \quad \longleftrightarrow \quad f(x, y) = -\gamma,$$

it becomes a constant affine-linear function, and there are two fundamentally different cases.

If $\gamma = 0$, then $f = 0$ and

$$Z(f) = \mathbb{R}^2.$$

The equation $0 = 0$ is a tautology and imposes no condition. Its augmented row is the zero vector $(0, 0, 0)$.

If $\gamma \neq 0$, then $f = -\gamma$ is a nonzero constant and

$$Z(f) = \varnothing.$$

The equation $0 = \gamma$ is a contradiction. Its augmented row $(0, 0, \gamma)$ is nonzero even though its first two entries vanish. Thus the augmented coordinate is essential:

> $(0, 0 \mid 0)$ means no condition, whereas $(0, 0 \mid \gamma)$, $\quad \gamma \neq 0$, means impossibility.

If either equation in a two-equation system has coefficient pair $(0, 0)$, then the coefficient matrix A is singular. Such a system can therefore never have a unique affine solution. The elementary possibilities are summarised below.

Equations	rank A	rank$[A \mid c]$	Solution set
One genuine line and $0 = 0$	1	1	The genuine line
One genuine line and $0 = \delta \neq 0$	1	2	$\varnothing$
$0 = 0$ **and** $0 = 0$	0	0	$\mathbb{R}^2$
Two constant rows; at least one is nonzero	0	1	$\varnothing$

This is the rank criterion

$$Ax = c \text{ is consistent} \quad \iff \quad \operatorname{rank} A = \operatorname{rank}[A \mid c].$$

For a consistent system the dimension of the solution set is $2 - \operatorname{rank} A$: rank 0 gives the whole plane, rank 1 a line, and rank 2 a point.

8.2 Redundancy and contradiction can be hidden

Consider first one genuine line together with a tautology:

$$M = \begin{bmatrix} \alpha & \beta & \eta \\ 0 & 0 & 0 \end{bmatrix}, \qquad (\alpha, \beta) \neq (0, 0).$$

The solution set is the line $\alpha x + \beta y = \eta$, and the RREF contains one nonzero row and one zero row. Nevertheless, the appearance of a zero row is not preserved by arbitrary row operations. For example,

$$\begin{bmatrix} \alpha & \beta & \eta \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} \alpha & \beta & \eta \\ \alpha & \beta & \eta \end{bmatrix}.$$

Both rows now look like genuine equations, but they describe the same line. The invariant statement is that the augmented row space has dimension 1; RREF makes the redundancy visible again.

Now consider one genuine line together with a contradiction:

$$M = \begin{bmatrix} \alpha & \beta & \eta \\ 0 & 0 & \delta \end{bmatrix}, \qquad (\alpha, \beta) \neq (0, 0), \quad \delta \neq 0.$$

After scaling the second row and using it to clear the last entry of the first row, one obtains

$$\begin{bmatrix} \alpha & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Normalising the genuine line equation gives one of the two rank-two inconsistent RREF forms listed in Section 1. The row [0 0 | 1] is the standard certificate of inconsistency.

Again, the certificate need not be visible in every row-equivalent presentation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The equations on the right are $x = 0$ and $x = 1$. The contradiction row has disappeared from sight, but the augmented row space still contains the nonzero constant equation $0 = 1$.

Finally, if both coefficient pairs vanish, the system is

$$0 = \gamma_1, \qquad 0 = \gamma_2.$$

It has solution set $\mathbb{R}^2$ when $\gamma_1 = \gamma_2 = 0$, and solution set $\varnothing$ otherwise. Thus two degenerate equations can describe the whole plane or the empty set, but never a line or a point.

8.3 Classification by the affine equation space

Let$V = \operatorname{span}\{f_1, f_2\} \subseteq \mathcal{A}$ be the affine equation space of the system. The common zero set is determined as follows.

Proposition 8.1. *For a system of two affine-linear equations in two variables,*

Condition on V	$Z(f_1) \cap Z(f_2)$
$V = \{0\}$	$\mathbb{R}^2$
$\dim V = 1,\ 1 \notin V$	*an affine line*
$\dim V = 2,\ 1 \notin V$	*one point*
$1 \in V$	$\varnothing$

Proof. If $V = 0$, both equations are zero functions. If $1 \in V$, a linear combination of the equations is a nonzero constant, so no point can satisfy both equations. If V is one-dimensional and contains no nonzero constant, it is spanned by a genuine affine-linear equation and its common zero set is a line.

Suppose finally that $\dim V = 2$ and $1 \notin V$. Let g_1, g_2 be a basis, with linear parts ℓ_1, ℓ_2. If these linear parts were dependent, say $\ell_2 = \lambda \ell_1$, then $g_2 - \lambda g_1$ would be constant. It would be nonzero because g_1, g_2 are independent, contradicting $1 \notin V$. Hence ℓ_1, ℓ_2 are independent, and the two basis equations meet at exactly one point. □

This includes the familiar singular cases in which both equations are genuine lines. If $\Delta = 0$, coincident lines give a one-dimensional V, while distinct parallel lines span a two-dimensional space containing 1. Thus consistency is detected not by the appearance of the individual rows but by whether their span contains a nonzero constant.

More explicitly, suppose both coefficient pairs are nonzero, $\Delta = 0$, and

$$(\alpha_2, \beta_2) = \lambda(\alpha_1, \beta_1).$$

Then the linear terms cancel in the combination

$$\lambda f_1 - f_2 = \gamma_2 - \lambda \gamma_1. \tag{17}$$

If the right-hand side is zero, the two equations are proportional and describe the same line. If it is nonzero, the lines are distinct and parallel, and equation (17) exhibits the constant contradiction contained in their equation space.

For a nonempty solution set S, it is useful to write

$$V_S = \{f \in \mathcal{A} : f|_S = 0\}.$$

If $S = L$ is a line, then V_L is one-dimensional and is spanned by any equation defining L. If $S = \mathbb{R}^2$, then $V_S = 0$. For a point $S = \{P\}$, this is the two-dimensional space V_P studied above. If $S = \varnothing$, however, every affine-linear function vanishes on S vacuously, so $V_\varnothing = \mathcal{A}$. A two-row space need not equal $\mathcal{A}$; this is why the identification of the row space with all affine-linear functions vanishing on the solution set requires consistency.

SECTION 9

The orbit space is a projective plane

We now assemble all ranks and consistency types into one orbit space. The lower-rank strata explain the stabilisers, while the rank-two stratum becomes the projective completion of the ordinary solution plane.

9.1 Orbits and stabilisers in every rank

The action

$$E \cdot M = EM, \qquad E \in \mathrm{GL}_2(\mathbb{R}),$$

continues to make sense for every 2×3 matrix, and elementary row matrices still generate $\mathrm{GL}_2(\mathbb{R})$. What changes in the degenerate cases is the geometry of the orbit.

Proposition 9.1. *Two matrices $M, N \in \mathbb{R}^{2\times 3}$ lie in the same $\mathrm{GL}_2(\mathbb{R})$-orbit if and only if*

$$\operatorname{Row} M = \operatorname{Row} N.$$

The action on such an orbit is free exactly when the common row rank is 2.

Proof. Left multiplication by an invertible matrix changes the ordered generating rows but not their span. Conversely, if the common row space has dimension 2, the rows of M and N are two ordered bases of that space, so a unique invertible change-of-basis matrix sends one pair to the other.

If the common row space has dimension 1, choose a nonzero row vector r spanning it and write

$$M = ur, \qquad N = vr,$$

for nonzero column vectors $u, v \in \mathbb{R}^2$. The group $\mathrm{GL}_2(\mathbb{R})$ acts transitively on the nonzero vectors of $\mathbb{R}^2$, so some $E \in \mathrm{GL}_2(\mathbb{R})$ satisfies $Eu = v$, and hence $EM = N$. The rank-zero case is immediate.

For freeness, a rank-two matrix has a right inverse, so $EM = M$ implies $E = I_2$. If

$$M = \begin{bmatrix} r \\ 0 \end{bmatrix}, \qquad r \neq 0,$$

then every matrix

$$E = \begin{bmatrix} 1 & s \\ 0 & t \end{bmatrix}, \qquad t \neq 0,$$

fixes M. The zero matrix is fixed by all of $\mathrm{GL}_2(\mathbb{R})$. ☐

The rank-two stratum contains both consistent systems with a unique solution and inconsistent systems whose row space contains a nonzero constant. Thus full augmented row rank alone does not imply a unique solution; one needs

$$\operatorname{rank} A = \operatorname{rank}[A \mid c] = 2.$$

The full orbit space is therefore stratified by

$$\mathrm{Gr}(0,3) \sqcup \mathrm{Gr}(1,3) \sqcup \mathrm{Gr}(2,3).$$

For consistent systems, the solution set determines the row space: the whole plane corresponds to the zero row space, a line to its one-dimensional equation space, and a point P to W_P. For inconsistent systems, the empty solution set is not a complete invariant. For example,

$$M_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are both inconsistent, but have augmented ranks 1 and 2, respectively, and hence cannot be row-equivalent.

9.2 Parallel families and the projective line at infinity

Homogenising the affine equation
$$\alpha x + \beta y = \gamma$$
gives the projective equation
$$\alpha X + \beta Y - \gamma Z = 0.$$
When $(\alpha, \beta) = (0, 0)$ and $\gamma \neq 0$, this reduces to $Z = 0$, the line at infinity in the projective plane. Thus the contradiction $0 = \gamma$ has no affine points, but its homogenisation is the line at infinity. By contrast, $0 = 0$ has all three coefficients zero and therefore has no projective class at all.

This also clarifies the geometry of distinct parallel lines. If $f = 0$ is one affine line, then the projectivisation of
$$\operatorname{span}\{f, 1\}$$
is the pencil of all lines parallel to $f = 0$, together with the line at infinity. These projective lines meet at the ideal point representing their common direction. Hence a two-dimensional equation space has one of two projective geometries:
$$\begin{array}{r|c} 1 \notin V & \text{a pencil through a finite point} \\ 1 \in V & \text{a pencil through a point at infinity.} \end{array}$$
The first is the unique-intersection case; the second is the inconsistent parallel case, in which row reduction reveals the affine contradiction $0 = 1$.

9.3 Grassmannian duality and the projective completion

Use affine-function coordinates
$$(u, v, w) \longleftrightarrow ux + vy + w$$
on $\mathcal{A}$. The isomorphism Φ identifies coefficient row spaces $W \subset \mathbb{R}^3$ with function spaces $V = \Phi(W) \subset \mathcal{A}$. Grassmannian duality sends a two-plane to the projective class of its one-dimensional annihilator:
$$\operatorname{Gr}(2, \mathcal{A}) \longrightarrow \mathbb{P}(\mathcal{A}^*), \qquad V \longmapsto [\operatorname{Ann}(V)].$$

Proposition 9.2. *Under Grassmannian duality, the rank-two orbit space is canonically $\mathbb{P}(\mathcal{A}^*) \cong \mathbb{RP}^2$. More precisely:*

(i) a consistent system with solution $P = (a, b)$ maps to the finite projective point $[a : b : 1]$;

(ii) an inconsistent rank-two system with parallel direction $(-v, u)$ maps to the ideal point $[-v : u : 0]$.

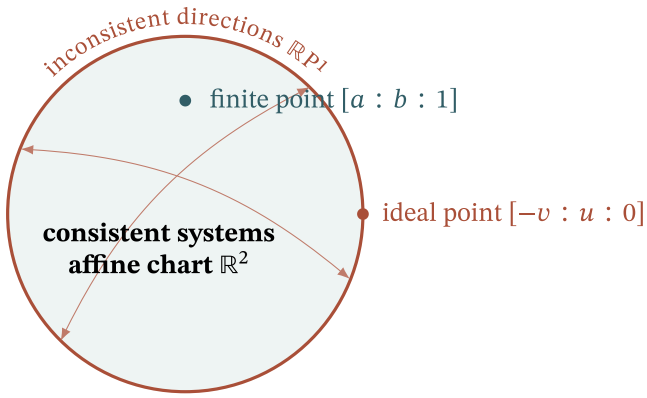
Thus $\mathbb{RP}^2$ is literally the projective completion of the affine solution plane.

Proof. For a finite point $P = (a, b)$, evaluation at P is the covector
$$\lambda_P(u, v, w) = ua + vb + w,$$
whose coordinate vector is $(a, b, 1)$. Since $V_P = \ker \lambda_P$, duality sends V_P to $[a : b : 1]$.

If V is inconsistent and two-dimensional, Proposition 8.1 gives $1 \in V$. Write
$$V = \operatorname{span}\{ux + vy + w, 1\}, \qquad (u, v) \neq (0, 0).$$
A covector (A, B, C) annihilating $1 = (0, 0, 1)$ must have $C = 0$; annihilating (u, v, w) then gives $uA + vB = 0$. Hence its projective class is $[-v : u : 0]$, the ideal point in the direction $(-v, u)$ of the parallel family.

By Proposition 8.1, every two-dimensional V falls under exactly one of (i) and (ii). The correspondence is therefore a bijection onto $\mathbb{P}(\mathcal{A}^*)$. ☐

Figure 3 shows the resulting orbit space, with finite solution points in the affine chart and inconsistent parallel directions on its boundary.



Disc model of $\mathbb{RP}^2$: antipodal boundary points are identified.

Figure 3: The rank-two orbit space is the projective completion of the solution plane. The interior is the affine chart of finite solutions; the antipodally identified boundary is the line of ideal parallel directions.

The RREF representatives make the three pivot-pattern strata explicit. The pivot pattern $(1, 2)$ gives

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix} \longmapsto [a : b : 1].$$

The other two patterns give

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \longmapsto [-a : 1 : 0], \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longmapsto [1 : 0 : 0].$$

These give a decomposition of $\mathbb{RP}^2$ into pieces parametrised by $\mathbb{R}^2$, $\mathbb{R}^1$, and $\mathbb{R}^0$, respectively.

Remark 9.3 (The finite-field shadow). Over $\mathbb{F}_2$, the rank-two orbit space has

$$|\mathrm{Gr}(2, 3)(\mathbb{F}_2)| = 2^2 + 2 + 1 = 7$$

points and is the Fano plane. Its three pivot-pattern subsets contain 4, 2, and 1 points. This numerical seven is distinct from the seven RREF *shapes* of Lemma 1.3, which range over all ranks.

SECTION 10

The general system and its planar origins

The preceding sections developed the geometry of two equations in two variables. We now pass to an arbitrary system, but the results below are not separate phenomena: each is a direct generalisation of a feature already visible in the planar discussion.

> **Five generalisations of the preceding sections.**
>
> 1. The contradiction $0 = 1$ and the constant consequences of Section 8 become a general criterion for inconsistency.
> 2. The identity $\Phi(\mathrm{Row}\, M) = V_P$ from Section 5 becomes a theorem identifying all affine-linear equations vanishing on the full solution set.
> 3. The horizontal and vertical members selected in Section 2 become the pivot equations of RREF, adapted to the ordered coordinate flag.
> 4. The coordinate lines $x = a$, $y = b$ from Sections 2 and 5 become the coordinate hyperplanes $x_i = p_i$ of an invertible square system.
> 5. The projective-plane orbit space of Section 9 becomes a disjoint union of Grassmannians, one for each possible row rank.

Let

$$Ax = c, \qquad A \in \mathbb{R}^{m\times n}, \quad c \in \mathbb{R}^m,$$

and let $\mathcal{A}_n$ denote the $(n+1)$-dimensional vector space of affine-linear functions on $\mathbb{R}^n$. The higher-dimensional analogue of Φ is the linear isomorphism

$$\Phi_n : \mathbb{R}^{n+1} \longrightarrow \mathcal{A}_n, \qquad \Phi_n(u,\gamma)(x) = u^{\mathrm{T}}x - \gamma. \tag{18}$$

If the i-th row of $[A \mid c]$ is (a_i^{T}, c_i), write

$$f_i(x) = a_i^{\mathrm{T}}x - c_i, \qquad V = \operatorname{span}\{f_1, \dots, f_m\} = \Phi_n(\operatorname{Row}[A \mid c]).$$

10.1 Row-space orbits in arbitrary size

The projective-plane picture of the preceding section is the first nontrivial case of a general orbit theorem.

Theorem 10.1 (GL$_m$-orbits are row spaces)**.** *For $M, N \in \mathbb{R}^{m\times(n+1)}$, there exists $E \in \mathrm{GL}_m(\mathbb{R})$ with*

$$N = EM$$

if and only if

$$\operatorname{Row} N = \operatorname{Row} M.$$

Consequently, the orbit space is the rank-stratified union

$$\bigsqcup_{r=0}^{\min\{m,n+1\}} \mathrm{Gr}(r, n+1).$$

Proof. Left multiplication by an invertible matrix changes the generating rows but not their span. Conversely, suppose the common row space W has dimension r, and let $B \in \mathbb{R}^{r\times(n+1)}$ have a basis of W as its rows. Then

$$M = CB, \qquad N = DB,$$

for matrices $C, D \in \mathbb{R}^{m\times r}$ of column rank r. Extend the columns of C and D to bases of $\mathbb{R}^m$. The invertible linear map carrying the first extended basis to the second has a matrix $E \in \mathrm{GL}_m(\mathbb{R})$ satisfying $EC = D$. Hence

$$EM = ECB = DB = N.$$

□

The standard uniqueness theorem for RREF [4] selects one representative of each orbit. Its pivot pattern records the successive rank jumps relative to the ordered coordinate flag, while its nonpivot entries supply the free parameters of the reduced representative. The action is free on the full-row-rank stratum $r = m$; lower-rank generating tuples have nontrivial stabilisers.

10.2 Constant contradictions and consistency

The first result generalises the planar distinction between a tautological row $0 = 0$ and a contradictory row $0 = \gamma \neq 0$, as well as the constant consequence produced by two distinct parallel lines.

Theorem 10.2 (Equivalent forms of inconsistency)**.** *For an arbitrary real system $Ax = c$, the following conditions are equivalent:*

(i) *the system is inconsistent;*

(ii) *V contains a nonzero constant function;*

(iii) $1 \in V$*;*

(iv) $\operatorname{rank}[A \mid c] > \operatorname{rank} A$*;*

(v) $\operatorname{rref}[A \mid c]$ *contains a row* $[0 \ \cdots \ 0 \mid 1]$.

Proof. In RREF, the augmented rank exceeds the coefficient rank exactly when the final column is a pivot column. Because it is the last column, its pivot row is

$$[0 \; \cdots \; 0 \mid 1],$$

which is the impossible equation $0 = 1$. Conversely, if the RREF contains no such row, arbitrary values may be assigned to the free variables and the pivot equations then determine a solution. This proves the equivalence of (i), (iv), and (v).

To connect rank with the equation space, project the augmented row space onto its first n coordinates:

$$\pi : \operatorname{Row}[A \mid c] \longrightarrow \operatorname{Row} A.$$

This map is onto, and its kernel consists exactly of the augmented vectors

$$(0, \dots, 0, \eta) \in \operatorname{Row}[A \mid c].$$

Rank–nullity gives

$$\dim \ker \pi = \operatorname{rank}[A \mid c] - \operatorname{rank} A.$$

Under Φ_n, a nonzero vector in this kernel represents the nonzero constant function $-\eta$. Thus (ii) and (iv) are equivalent. Finally, any nonzero constant can be scaled to 1, so (ii) and (iii) are equivalent. $\square$

10.3 The complete space of affine consequences

The next theorem generalises the point equation space V_P. In lower rank, the solution set may instead be an affine line, plane, or higher-dimensional affine subspace, but the image of the augmented row space under Φ_n continues to contain exactly its affine-linear consequences.

Assume from now on that the solution set

$$S = \{x \in \mathbb{R}^n : Ax = c\}$$

is nonempty.

Theorem 10.3 (Row space as all affine-linear equations vanishing on the solution set)**.** *For a consistent system,*

$$V = \{f : \mathbb{R}^n \to \mathbb{R} \text{ affine-linear} : f|_S = 0\}. \tag{19}$$

In particular, $\dim V = \operatorname{rank} A$.

Proof. Every linear combination of the original equations vanishes on every solution, so V is contained in the right-hand side.

Conversely, choose $x_0 \in S$. Then $S = x_0 + \ker A$. Suppose $f(x) = u^{\mathrm{T}}x - v$ vanishes on S. For every $z \in \ker A$,

$$0 = f(x_0 + z) - f(x_0) = u^{\mathrm{T}}z.$$

Thus u annihilates $\ker A$, so u^{T} lies in the row space of A. Hence $u^{\mathrm{T}} = \lambda^{\mathrm{T}}A$ for some $\lambda \in \mathbb{R}^m$. Since $f(x_0) = 0$,

$$v = u^{\mathrm{T}}x_0 = \lambda^{\mathrm{T}}Ax_0 = \lambda^{\mathrm{T}}c.$$

Therefore

$$f(x) = \lambda^{\mathrm{T}}(Ax - c) = \sum_i \lambda_i f_i(x),$$

so $f \in V$. Finally, consistency gives $\operatorname{rank}[A \mid c] = \operatorname{rank} A$; since the row-to-function map is injective, $\dim V = \operatorname{rank} A$. $\square$

10.4 The echelon staircase and the coordinate flag

In two variables, cancellation selected the unique horizontal and vertical members of the pencil through the solution point. For a general system, the same coordinate dependence is encoded by the pivot columns and the ordered coordinate flag.

The graph structure of the solution set and the staircase of pivot positions encode related but distinct facts. For $j = 1, \dots, n$, let $A_{[j]}$ denote the matrix formed by the first j columns of A, and let $A_{[0]}$ be the empty matrix.

Proposition 10.4 (Pivot columns and the coordinate flag)**.** *For a consistent system, variable column j is a pivot column of* rref$[A \mid c]$ *exactly when*

$$\operatorname{rank} A_{[j]} > \operatorname{rank} A_{[j-1]}. \tag{20}$$

Thus the pivot indices are determined by A together with the ordered coordinate flag

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^n.$$

Proof. Row operations preserve the rank of every initial column block. In RREF, the rank of the first j variable columns is the number of pivots among those columns. It therefore increases at j precisely when column j is a pivot column. Consistency ensures that the augmented column is not an additional pivot column. □

Figure 4 shows a schematic echelon staircase and the associated free entries.

RREF gives the unique reduced basis of the affine equation space V adapted to this ordered coordinate structure. If the pivot variables are $x_{j_1}, \ldots, x_{j_r}$ and the remaining coordinates x_k are free, the nonzero RREF equations have the form

$$x_{j_i} = d_i - \sum_{k \text{ free}} r_{ik} x_k, \qquad i = 1, \ldots, r. \tag{21}$$

Thus projection onto the free-coordinate subspace is an affine isomorphism from S to $\mathbb{R}^{n-r}$. This explains why the solution set is presented as a graph. Proposition 10.4 explains separately why the leading entries form an echelon staircase: the staircase records the successive rank jumps obtained by projecting the coefficient row space onto the nested initial-coordinate subspaces.

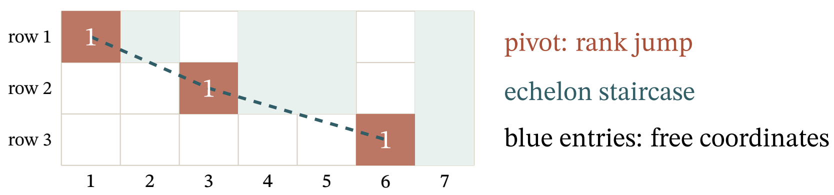


Figure 4: A schematic RREF staircase for pivot columns $1, 3, 6$. Each pivot marks a rank jump along the ordered coordinate flag; the entries to the right that are not in later pivot columns are the free parameters of the reduced basis. The seventh column may be read as the augmented column; its nonpivot status records consistency.

10.5 The square full-rank case

The final statement generalises the original picture of two intersecting lines. In $\mathbb{R}^n$, the coordinate lines $x = a$, $y = b$ are replaced by the n coordinate hyperplanes through the solution point.

Corollary 10.5 (RREF of an invertible square system)**.** *Suppose $m = n$. The system has a unique solution exactly when A is invertible. If its solution is $P = (p_1, \ldots, p_n)$, then*

$$\operatorname{rref}[A \mid c] = [I_n \mid P],$$

and the output equations $x_i = p_i$ are the n coordinate hyperplanes through P.

Proof. If A is invertible, then $P = A^{-1}c$ is the unique solution and left multiplication by A^{-1} gives

$$A^{-1}[A \mid c] = [I_n \mid P],$$

which is already in RREF. Conversely, if a square system has a unique solution, then $\ker A = \{0\}$, so A is invertible. The two-line theorem is the case $n = 2$. □

Geometric dictionary

Algebraic object or operation	Geometric meaning
Augmented row (α, β, γ)	Affine function $\alpha x + \beta y - \gamma$; its zero set is a line when $(\alpha, \beta) \neq 0$.
Row space of $[A \mid c]$	All affine-linear consequences of the equations; for a consistent system, all affine-linear functions vanishing on the solution set.
$1 \in V$	A constant contradiction lies in the affine-function space; the system has no solution.
$e_{n+1} \in W$	The coefficient row space contains a constant row; equivalently $1 \in V = \Phi_n(W)$, up to sign.
RREF row $[0 \ \cdots \ 0 \mid 1]$	Canonical certificate of inconsistency; projectively, the line at infinity occurs as a consequence.
Row interchange	Relabel equations; choose a line capable of the required cancellation.
Nonzero row scaling	Choose a different vector representative of the same geometric line.
$R_2 \mapsto R_2 + tR_1$	Replace one line by another member of the pencil through the common point.
Transvection in coefficient space	Translation along the affine line $r_2 + tr_1$ inside the fixed row space.
Cancel the x**-coefficient**	Select the horizontal member $y = b$.
Cancel the y**-coefficient**	Select the vertical member $x = a$.
RREF in the full-rank 2×2 **case**	Canonical basis $x - a, y - b$ of V_P; it remembers exactly P.
GL_m**-orbit of** $[A \mid c]$	The augmented row space; a point of the appropriate Grassmannian.
V_S	All affine-linear consequences vanishing on the nonempty solution set S.
RREF pivot pattern	The positions of successive rank jumps relative to the ordered coordinate flag.
Pivot column	A coordinate solved as an affine function of the free coordinates.
Free column	A coordinate used to parametrise the solution affine subspace.
Rank-two row space in the affine chart of $\mathrm{Gr}(2, 3)$	A consistent system, parametrised by its unique solution point.
Rank-two row space on the line at infinity	An inconsistent system, parametrised by the common direction of its parallel family.

Notes and further reading

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